RIMS Kôkyûroku Bessatsu
B62 (2017), 53–70

Conformal field theory for C_2-cofinite VOAs and fusion tensor products

By

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Abstract

In this talk we announce the factorization theorem in conformal field theory in two-dimension for C_2-cofinite VOAs, which describes the behavior of conformal blocks under degeneration of Riemann surfaces. As an application we discuss the fusion tensor product of modules of a VOA.

§ 1. Introduction

This talk is based on a project with Professor Akihiro Tsuchiya.

Conformal field theory (CFT) is a quantum field theory (in two-dimension) invariant under conformal transformations. In mathematics 2D CFT are formulated in the language of algebraic geometry of algebraic curves (Riemann surfaces) and D-modules.

First we show a rough sketch. Let $\pi : C = \bigcup_{t\in S} C_t \to S$ is a family of compact Riemann surfaces of genus g over a complex manifold S with m disjoint sections $s_\alpha : S \to C\,(\alpha \in \Lambda = \{1, \ldots, m\})$.

We consider "fields" $A_i(z_i)$ $(i \in I = \{1, \ldots, n\})$ located at $z_i \in C_t$ and "states" $\Omega_\alpha(w_\alpha)$ $(\alpha \in \Lambda)$ located at $w_\alpha = s_\alpha(t) \in C_t$, where z_i, w_α are local coordinates parametrizing points on C_t. Let the parameters $(w_\Lambda, t_\Gamma) = (w_1, \ldots, w_m;\ t_1, \ldots, t_l)$ be a local coordinate of B.

The fundamental quantity which describes the quantum system is a system of "n-point functions"

$$\langle \Phi_n(t_\Gamma) \mid A_1(z_1) \cdots A_n(z_n) \mid \Omega_1(w_1) \otimes \cdots \otimes \Omega_m(w_m)\rangle, \quad n = 0, 1, 2, \ldots,$$

each of which is locally a meromorphic function of $z_I = (z_1, \ldots, z_n), w_\Lambda, t_\Gamma$ with poles along $z_i = z_j\ (i \neq j)$, $z_i = w_\alpha$, invariant under the permutation of I.

Received September 23, 2015. Revised January 5, 2017.
2010 Mathematics Subject Classification(s): 14H70, 17B69, 81T40
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The linear functional $\langle \Phi_n \mid$ gives a global meromorphic section of a vector bundle over $\bigcup_{t \in S} (C_t)^n$.

The *conformal block* is a 0-point function with no obstruction to the existence of a system of "n-point functions."

The behavior of the "n-point function" under the differentiation $\dfrac{\partial}{\partial z_i}$ and the limit $z_i \to z_j \, (i \neq j)$ is controlled by a *vertex operator algebra (VOA) V*, and the behavior under $z_i \to w_\alpha$ is controlled by a *V-module* given for each α.

Under suitable assumptions it is shown that the n-point functions satisfy the *Knizhnik-Zamolodchikov (KZ)-equations* which describe the behavior under $\dfrac{\partial}{\partial w_\alpha}$ or $\dfrac{\partial}{\partial t_\gamma}$.

Thus if we perceive the asymptotic behavior under $w_\alpha \to w_\beta \, (\alpha \neq \beta)$ and $t \to t_0$ where C_{t_0} is a singular curve, then it is expected that we can determine the n-point functions in detail. The asymptotic behavior is described by the "*factorization theorem.*"

In the late 1980s Professor Tsuchiya and his collaborators achieved a great success in this project for the case associated to integrable highest weight representations of simple affine Lie algebras [8], [9], [10]. It corresponds to the case that the abelian category of V-modules is semi-simple [7].

In this talk we announce the factorization theorem for the "*logarithmic*" CFT, the case in which the abelian category of V-modules is not semi-simple. As an application we discuss the "*fusion tensor products*" of V-modules [2]. More detailed discussion shall be given in the forthcoming paper with A. Tsuchiya and T. Matsumoto.

§ 2. Vertex algebras and VOAs

In regard to this section see [1], [5].

Definition 2.1. Let V be a vector space over $\mathbb{C}$, $|0\rangle \in V$ and $L_{-1} : V \to V$ a linear map satisfying $L_{-1}|0\rangle = 0$. We consider a linear map

$$Y : V \otimes V \to V(\!(z)\!) = V[[z]] \oplus z^{-1} V[z^{-1}],$$

$$Y(a \otimes b) = Y(a, z)\, b = \sum_{n \in \mathbb{Z}} a_{(n)}\, b\, z^{-n-1}.$$

satisfying $\dfrac{\partial}{\partial z} Y(a, z) = [L_{-1}, Y(a, z)]$, $Y(a, z)|0\rangle = \mathrm{e}^{z L_{-1}} a$, and the *locality condition*:

For any $a, b \in V$ there exists $N \in \mathbb{N}$ such that

$$(z - w)^N \, Y(a, z)\, Y(b, w) = (z - w)^N \, Y(b, w)\, Y(a, z).$$

We call V a *vertex algebra.*

Proposition 2.2. *(1) For any $a,\, b,\, c \in V$ there exists*

$$Y(a,\, z) \circ Y(b,\, w)|c\rangle \in w^{-N_1}\, z^{-N_2}\, (z-w)^{-N_3}\, V[[z,\, w]]$$

$(N_1,\, N_2,\, N_3 \in \mathbb{N})$ such that

- $Y(a,\, z) \circ Y(b,\, w)|c\rangle|_{|z|>|w|} = Y(a,\, z)\, Y(b,\, w)\, c$ *where* $\left.\dfrac{1}{z-w}\right|_{|z|>|w|} = \dfrac{1}{z}\left(1 + \dfrac{w}{z} + \dfrac{w^2}{z^2} + \cdots\right)$
- $Y(a,\, z) \circ Y(b,\, w)|c\rangle|_{|w|>|z|} = Y(b,\, w)\, Y(a,\, z)\, c$ *where* $\left.\dfrac{1}{z-w}\right|_{|w|>|z|} = -\dfrac{1}{w}\left(1 + \dfrac{z}{w} + \dfrac{z^2}{w^2} + \cdots\right)$
- *Operator Product Expansion (OPE)* $Y(a,\, z) \circ Y(b,\, w)|c\rangle|_{|w|>|z-w|} = Y(Y(a,\, z-w)\, b,\, w)\, c$ *where* $\left.\dfrac{1}{z}\right|_{|w|>|z-w|} = \dfrac{1}{w}\left(1 + \dfrac{w-z}{w} + \dfrac{(w-z)^2}{w^2} + \cdots\right).$

(1') The Borcherds identity: For any $a,\, b \in V,\ p.\, q,\, r \in \mathbb{Z}$

$$\begin{aligned}&\sum_{i\geq 0} \binom{p}{i} (a_{(r+i)}\, b)_{(p+q-i)} \\ &= \sum_{j \geq 0} (-1)^j \binom{r}{j} \left(a_{(p+r-j)}\, b_{(q+j)} - (-1)^r b_{(q+r-j)}\, a_{(p+j)}\right).\end{aligned}$$

(Remark that the LHS has finite terms and the RHS has infinite terms.)

(2) $\quad \dfrac{\partial}{\partial z} Y(a,\, z) = Y(L_{-1}a,\, z), \quad Y(|0\rangle,\, z) = \mathrm{id}_V.$

Definition 2.3. A vector space M over $\mathbb{C}$ together with a linear map

$$Y^M : V \otimes M \to M(\!(z)\!), \quad Y^M(a \otimes \Omega) = Y^M(a,\, z)\, \Omega$$

is called a *V-module* if

1. For any $a,\, b \in V,\ \Omega \in M$ there exists

$$Y^M(a,\, z) \circ Y^M(b,\, w)|\Omega\rangle \in w^{-N_1}\, z^{-N_2}\, (z-w)^{-N_3}\, M[[z,\, w]]$$

   $(N_1,\, N_2,\, N_3 \in \mathbb{N})$ such that

   - $Y^M(a,\, z) \circ Y^M(b,\, w)|\Omega\rangle\big|_{|z|>|w|} = Y^M(a,\, z)\, Y^M(b,\, w)\, \Omega$

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- $Y^M(a,\, z) \circ Y^M(b,\, w)|\Omega\rangle\big|_{|w|>|z-w|} = Y^M(Y(a,\, z-w)\, b,\, w)\, \Omega$

2. $\dfrac{\partial}{\partial z} Y^M(a,\, z) = Y^M(L_{-1}a,\, z), \quad Y^M(|0\rangle,\, z) = \mathrm{id}_M.$

A vertex algebra V itself is naturally considered to be a V-module. It is called *vacuum module.*

Proposition 2.4. *Let* $\mathrm{Lie}\,(V) = (V \otimes \mathbb{C}[t])/\mathrm{Im}\,(\partial_t + L_{-1})$. *It has a structure of a Lie algebra defined by the Borcherds identities*

$$[a_{(p)},\, b_{(q)}] = \sum_{i \geq 0} \binom{p}{i} (a_{(i)}\, b)_{(p+q-i)} \quad (a_{(p)} = [a \otimes t^p]).$$

A V-module is naturally considered to be a $\mathrm{Lie}\,(V)$-module.

For a vertex algebra V and $n \geq 2$ we denote by C_nV the vector subspace spanned by $a_{(-n)}\, b$ $(a,\, b \in V)$. It holds $C_nV \supset C_{n+1}V$.

Proposition 2.5. *It holds* $\dim V/C_nV < \infty$ *if and only if* $\dim V/C_{n+1}V < \infty$.

Definition 2.6. A vertex algebra V is called C_2*-cofinite* if $\dim V/C_2V < \infty$.

Definition 2.7. The Lie algebra $\mathrm{Vir} = \left(\bigoplus_{n \in \mathbb{Z}} \mathbb{C}\, L_n\right) \oplus \mathbb{C}\, C$ defined by

$$[L_m,\, L_n] = (m-n)\, L_{m+n} + \frac{m^3 - m}{12}\, \delta_{m+n,\, 0}\, C, \quad [C,\, L_n] = 0$$

is called *Virasoro algebra.*

It acts on $\mathbb{C}[z,\, z^{-1}]$ by $L_n = -z^{n+1}\, \partial_z,\ C = 0$. We denote by $\mathrm{Vir}_{\geq -1}$ the Lie subalebra spanned by L_n $(n \geq -1)$.

For $g \in \mathrm{Aut}\, \mathbb{C}[[z]],\ g \cdot z = \sum_{n=1}^{\infty} c_n\, z^n$ $(c_1 \neq 0)$ we can find $b_n \in \mathbb{C}$ such that

$$g \cdot z^k = \exp\left(-\sum_{n=1}^{\infty} b_n\, z^{n+1}\, \partial_z\right) {c_1}^{z\, \partial_z} \cdot z^k.$$

The group $\mathrm{Aut}\, \mathbb{C}[[z]]$ acts on a $\mathrm{Vir}_{\geq -1}$-module diagonalizable for L_0 with integral L_0-eigenvalues bounded below by

$$g \mapsto \rho(g) = \exp\left(\sum_{i=1}^{\infty} b_n\, L_n\right) {c_1}^{-L_0}.$$

Let $\mathrm{Aut}_0\, \mathbb{C}[[z]]$ be the subgroup consisting of the elements with $c_1 = 1$. It acts on a $\mathrm{Vir}_{\geq -1}$-module with L_0-eigenvalues bounded below.

Definition 2.8. (1) An element $|\mathrm{vir}\rangle$ of a vertex algebra V is called a *Virasoro element* of central charge $c \in \mathbb{C}$ if the coefficients of the Laurent expansion $Y(|\mathrm{vir}\rangle, z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2}$ satisfy

$$[L_m,\, L_n] = (m-n)\, L_{m+n} + \frac{m^3-m}{12}\,\delta_{m+n,\,0}\, c\cdot \mathrm{id}_V$$

where L_{-1} coincides with the one already given.
(2) $(V,\, |\mathrm{vir}\rangle)$ is called a *vertex operator algebra (VOA)* if

1. $V = \bigoplus_{\Delta=0}^{\infty} V[\Delta], \quad \dim V[\Delta] < \infty, \quad L_0\, a = \Delta\, a \quad (a \in V[\Delta])$
2. $V[0] = \mathbb{C}|0\rangle$.

For $a \in V[\Delta],\ n \in \mathbb{Z}$ we set $J_n(a) = a_{(\Delta+n-1)} \in \mathrm{Lie}\,(V)$. It holds $[L_0,\, J_n(a)] = -n\, J_n(a)$. We denote by $\mathrm{Lie}\,(V)[-n]$ the subspace spanned by operators $J_n(a)$ $(a \in V)$. Then

$$\mathrm{Lie}\,(V) = \bigoplus_{\Delta\in\mathbb{Z}} \mathrm{Lie}\,(V)[\Delta]$$

is a $\mathbb{Z}$-graded Lie algebra.

Proposition 2.9. *There exists an anti-automorphism θ on* $\mathrm{Lie}\,(V)$ *defined by*

$$\theta(J_n(a)) = J_{-n}(\mathrm{e}^{L_1}\,(-1)^{L_0}\, a).$$

Remark that

$$\mathrm{e}^{-z^2\partial_z}\,(-1)^{-z\partial_z}\cdot z = -\frac{z}{1+z},$$

$$w = -\frac{z}{1+z} \quad \Leftrightarrow \quad (1+z)(1+w) = 1.$$

§ 3. $\mathbb{Z}$-graded algebras associated to VOAs

In regard to this section see [6], [7].

We construct a $\mathbb{Z}$-graded associative algebra associated to a VOA satisfying the Borcherds idenity. Since the Borcherds idenity has infinite terms, we apply the theory of topological algebras.

Let $U = \bigoplus_{n\in\mathbb{Z}} U[n]$ be a $\mathbb{Z}$-graded algebra. We set

$$F_{\le p}U = \bigoplus_{n\le p} U[n], \quad F^{\ge p}U = \bigoplus_{n\ge p} U[n],$$

$$U^{\mathrm{L}} = \lim_{\leftarrow} U/F_{\leq -p-1}U, \quad U^{\mathrm{R}} = \lim_{\leftarrow} U/F^{\geq p+1}U.$$

For a left U-module M and a right U-module W we set

$$K_pM = \{u \in M \mid F_{\leq -p-1}U \cdot u = \{0\}\},\ K_p^{\vee}W = \{u \in W \mid u \cdot F^{\geq p+1}U = \{0\}\}.$$

For a U-bimodule X we set $X^{\mathrm{reg}} = \bigcup_{p,\,q} K_p^{\vee}K_qX$, which is a $(U^{\mathrm{L}},\, U^{\mathrm{R}})$-module.

Let $I_p[n] = I^{\vee}_{p+n}[n] = U[n] \cap U \cdot F_{\leq -p-1}U = U[n] \cap F^{\geq p+n+1}U \cdot U$, and

$$U[n]^{\wedge} = \lim_{\leftarrow} U[n]/I_q[n], \quad I_p[n]^{\wedge} = \lim_{\leftarrow} I_p[n]/I_{p+q}[n].$$

Let V be a VOA and $\widetilde{U}(V)$ be the quotient of the universal envelopping algebra of $\mathrm{Lie}\,(V)$ by the relation $Y(|0\rangle,\, z) = 1$. It has a natural $\mathbb{Z}$-grading $\widetilde{U}(V) = \bigoplus_{\Delta\in\mathbb{Z}} \widetilde{U}(V)[\Delta]$. We set $U(V)[\Delta]$ be the quotient of the completion $\widetilde{U}(V)[\Delta]^{\wedge}$ by the all Borcherds identities.

We call the direct sum $U(V) = \bigoplus_{\Delta\in\mathbb{Z}} U(V)[\Delta]$ the *current algebra* of V.

We call a V-module M *meromorphic* if

1. M is a finitely-generated left $U(V)$-module.
2. For any $u \in M$, there exists a positive integer n such that $F_{\leq -n}U(V) \cdot u = \{0\}$.
3. For any $u \in M$, $F_{\leq 0}U(V) \cdot u$ is of finite dimension.

Proposition 3.1. *(fermionic property [7])* $\mathrm{Lie}\,(V)[n] + I_p[n] = U(V)[n]$.

We call $U(V)^{\mathrm{reg}}$ the *regular bimodule.*

We define the $\mathbb{C}[X,\, Y]$-action on $U(V)$ by

$$X \cdot u = L_0\, u, \quad Y \cdot u = u\, L_0 \quad (u \in U(V)^{\mathrm{reg}}).$$

Proposition 3.2. *The anti-automorphism θ on* $\mathrm{Lie}\,(V)$ *can be extended to an anti-automorphism on $U(V)$.*

A right U^{R}-module can be considered to be a left U^{L}-module by way of θ.

Proposition 3.3. *([6]) Let be a C_2-cofinite VOA.*

1. *For each $u \in U(V)^{\mathrm{reg}}$ the vector space $\mathbb{C}[X,\, Y] \cdot u$ is of finite dimension.*
2. *$U(V)^{\mathrm{reg}}$ has a direct sum decomposition by generalized eigenvalues λ, μ for X, Y:*

$$U(V)^{\mathrm{reg}} = \bigoplus_{\lambda,\, \mu\in\Gamma} U[\lambda,\, \mu],$$

where $\Gamma = \Gamma_0 + \mathbb{Z}_{\geq 0}$, Γ_0 is a finite subset of $\mathbb{C}$ with no pairs of elements of integral differences, and each $U[\lambda,\, \mu]$ is of finite dimension.

The subalgebra $U[\lambda, \lambda] \subset U(V)$ contains the unity e_λ. We set

$$e_\infty(t) = \sum_{\lambda \in \Gamma} e_\lambda\, t^\lambda \in \bigoplus_{\mu \in \Gamma_0} U^{\mathrm{reg}}[[t]] \cdot t^\mu .$$

We call it *running unity.*

Proposition 3.4. *(1) For any $X \in U[\Delta]$ it holds*

$$\begin{aligned} &t^\Delta\, X \cdot e_\infty(t) = e_\infty(t) \cdot X \quad (\Delta \geq 0), \\ &X \cdot e_\infty(t) = t^{-\Delta}\, e_\infty(t) \cdot X \quad (\Delta < 0). \end{aligned}$$

(2) There exists $r \in \mathbb{N}$ such that $(t \dfrac{\mathrm{d}}{\mathrm{d}t} - L_0)^r\, e_\infty(t) = 0$.

We set $U_t = U(V) \otimes \mathbb{C}[[t]]$, $U_{(t)} = U(V) \otimes \mathbb{C}((t))$. We denote by B^∞ the U_t-bimodule generated by $e_\infty(t)$. For $n \in \mathbb{Z}_{\geq 0}$ we set

$$B^n = B^\infty \otimes_{\mathbb{C}[[t]]} (\mathbb{C}[[t]]/t^{n+1}), \quad e_n(t) = e_\infty(t) \otimes 1 \in B^n .$$

For $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ we set $B^m_{(t)} = B^m \otimes_{\mathbb{C}[[t]]} \mathbb{C}((t))$, which is a $U_{(t)}$-bimodule. It holds

$$\begin{aligned} B^m_{(t)} &= (\mathrm{Lie}\,(V) \otimes \mathbb{C}((t))) \cdot e_m(t) = e_m(t) \cdot (\mathrm{Lie}\,(V) \otimes \mathbb{C}((t))) \\ &= (U(V) \otimes \mathbb{C}((t))) \cdot e_m(t) = e_m(t) \cdot (U(V) \otimes \mathbb{C}((t))) \\ &= (U(V)^{\mathrm{L}} \otimes \mathbb{C}((t))) \cdot e_m(t) = e_m(t) \cdot (U(V)^{\mathrm{R}} \otimes \mathbb{C}((t))) \\ &= (U(V)^{\mathrm{reg}} \otimes \mathbb{C}((t))) \cdot e_m(t) = e_m(t) \cdot (U(V)^{\mathrm{reg}} \otimes \mathbb{C}((t))). \end{aligned}$$

§ 4. Covacua and conformal blocks

§ 4.1. Stable curves

Definition 4.1. ([3]) Let g, n be non-negative integers such that $2g-2+n > 0$. We set $\Lambda = \{1, 2, \ldots, n\}$.
(1) An n-pointed (or Λ-pointed) curve $(C,\ s_\Lambda = \{s_\alpha\}_{\alpha \in \Lambda})$ is *stable* if it is reduced with at most ordinary double points which are not marked, connected, and has only a finite number of automorphisms.
(2) A *stable n-pointed (or Λ-pointed) curve over a scheme S* is a proper flat morphism $\pi_C : C \to S$ of schemes together with n non-crossing sections $s_\Lambda = \{s_\alpha : S \to C\}_{\alpha \in \Lambda}$ not meeting any double points such that the geometric fibers of π_C are stable n-pointed curves.
(3) Let $\overline{\mathcal{M}}_{g,\,n}$ be the fibered category over the category of all schemes such that each

object is a stable n-pointed curve of genus g over a scheme. The morphisms are cartesian diagrams commuting with sections. The functor to the category of schemes takes an object $(\pi_C : C \to S,\ \ s_\Lambda)$ to its base S.
(4) Let $\overline{\mathcal{C}}_{g,\,n}$ be the fibered category over the category of all schemes such that each object $(\pi_C : C \to T,\ \{s_\alpha : T \to C\}_{\alpha\in\Lambda},\ \Delta)$ is a stable n-pointed curve of genus g over a scheme T with an extra section $\Delta : T \to C$.

The functor $\pi_{g,\,n} : \overline{\mathcal{C}}_{g,\,n} \to \overline{\mathcal{M}}_{g,\,n}$ defined by forgetting Δ is a morphism of fibered categories. For an object $\mathfrak{C} = (\pi_C : C \to S,\ \ s_\Lambda)$ of $\overline{\mathcal{M}}_{g,\,n}$, we take the first projection $\pi_{C\times_S C} = \mathrm{pr}_1 : C \times_S C \to C$, the sections $s^+_\Lambda = \{s^+_\alpha : C \to C \times_S C\}_{\alpha\in\Lambda}$ which are the pull-back's of s_Λ by π_C, and the diagonal morphism $\Delta_C : C \to C \times_S C$. Then we obtain an object of $\overline{\mathcal{C}}_{g,\,n}$

$$\mathfrak{C}^+ = (\pi_{C\times_S C} : C \times_S C \to C,\ s^+_\Lambda,\ \Delta_C).$$

For $\alpha \in \Lambda$, the pull-back ${s_\alpha}^{-1}\mathfrak{C}^+$ is an object of $\overline{\mathcal{C}}_{g,\,n}$. Then $\varphi \mapsto {s_\alpha}^{-1}\mathfrak{C}^+$ gives a morphism of fibered category $\sigma_\alpha : \overline{\mathcal{M}}_{g,\,n} \to \overline{\mathcal{C}}_{g,\,n}$.

Then $(\pi_{g,\,n} : \overline{\mathcal{C}}_{g,\,n} \to \overline{\mathcal{M}}_{g,\,n},\ \sigma_\Lambda = \{\sigma_\alpha\}_{\alpha\in\Lambda})$ is the universal stable n-pointed curve of genus g over $\overline{\mathcal{M}}_{g,\,n}$.

For a stable n-pointed curve $\mathfrak{C} = (\pi_C : C \to S,\ \ s_\Lambda)$ of genus g, a morphism $S \to \overline{\mathcal{M}}_{g,\,n}$ called *classifying morphism* and a morphism $C \to \overline{\mathcal{C}}_{g,\,n}$ are defined as follows.

The scheme S is identified with the slice category Scheme$/S$. For a morphism $\varphi : X \to S$ the pull-back of $\varphi^{-1}\mathfrak{C}$ gives a stable n-pointed curve over X. The correspondence $\varphi \mapsto \varphi^{-1}\mathfrak{C}$ gives the morphism $S \to \overline{\mathcal{M}}_{g,\,n}$.

Corresponding each morphism $\psi : Y \to C$ to the pull-back $\psi^{-1}\mathfrak{C}^+$, we obtain a morphism $C \to \overline{\mathcal{C}}_{g,\,n}$. The diagram

$$\begin{array}{ccc} C & \to & \overline{\mathcal{C}}_{g,\,n} \\ \downarrow & & \downarrow \\ S & \to & \overline{\mathcal{M}}_{g,\,n} \end{array}$$

is cartesian and commutes with sections.

F. Knudsen constructed a morphism $c : \overline{\mathcal{M}}_{g,\,n+1} \to \overline{\mathcal{C}}_{g,\,n}$ called *contraction* and a morphism $s : \overline{\mathcal{C}}_{g,\,n} \to \overline{\mathcal{M}}_{g,\,n+1}$ called *stabilization* in [3], [4].

Let $\Lambda = \{1,\,\ldots,\,n\}$, $\Lambda^+ = \Lambda \cup \{n+1\}$. Take a stable Λ^+-pointed curve $\mathfrak{C}' = (\pi_{C'} : C' \to S,\ s'_{\Lambda^+} = \{s'_\alpha\}_{\alpha\in\Lambda^+})$ of genus g. By forgetting s_{n+1}, we obtained a family of curves over S, but it is not a stable Λ-pointed curve in general.

Definition 4.2. Let $\mathfrak{C} = (\pi_C : C \to S,\ s_\Lambda = \{s_\alpha\}_{\alpha\in\Lambda},\ \Delta = s_{n+1})$ be an object of $\overline{\mathcal{C}}_{g,\,n}$. A morphism $\varphi : C' \to C$ over S commuting with sections $\{s'_\alpha\}_{\alpha\in\Lambda'}$, $\{s_\alpha\}_{\alpha\in\Lambda^+}$

is called *contraction* if each induced morphism between geometric fibers $\varphi_b : C'_b \to C_b$ is an isomorphism or there is a rational component $E \subset C'_b$ such that $s_{n+1}(b) \in E$, $\varphi_b(E) = \Delta(b)$ and $\varphi_b : C'_b \setminus E \to C_b \setminus \{\Delta(b)\}$ is an isomorphism.

For any object $\mathfrak{C}'$ of $\overline{\mathcal{M}}_{g,\,n+1}$, there is a contraction $\varphi : C' \to C$ unique up to isomorphisms. Then the correspondence $\mathfrak{C}' \mapsto \mathfrak{C}$ gives a morphism of fibered categories $c : \overline{\mathcal{M}}_{g,\,n+1} \to \overline{\mathcal{C}}_{g,\,n}$.

Definition 4.3. Let $\mathcal{J} \subset \mathcal{O}_C$ be the defining ideal (of the image) of Δ. We denote by $\mathcal{K}$ the cokernel of the diagonal morphism $\mathcal{O}_C \to \mathcal{J}^\vee \oplus \mathcal{O}_C(s_\Lambda)$, where s_Λ is the divisor on C given by the sections s_I. We set $C^{\mathrm{st}} = \mathrm{Proj}_C(\mathrm{Sym}\,\mathcal{K})$. There is a natural map $C^{\mathrm{st}} \to C$. A morphism $\pi_{C^{\mathrm{st}}} : C^{\mathrm{st}} \to S$ and sections $s'_\alpha(\alpha \in \Lambda)$, $\Delta' = s'_{n+1} : S \to C^{\mathrm{st}}$ are induced from $\mathfrak{C}$. Then $s(\mathfrak{C}) = (\pi_{C^{\mathrm{st}}} : C^{\mathrm{st}} \to S,\ s'_{\Lambda^+})$ is an object of $\overline{\mathcal{M}}_{g,\,n+1}$, called the *stabilization* of $\mathfrak{C}$. Thus we obtain a morphism of fibered categories $s : \overline{\mathcal{C}}_{g,\,n} \to \overline{\mathcal{M}}_{g,\,n+1}$.

Let $\mathfrak{C} = (\pi_C : C \to S,\ s_\Lambda)$ be a stable Λ-pointed curve over S with non-singular general fibers. Let $\mathfrak{C}^{(2)} = s(\mathfrak{C}^+) = (\pi_{C^{(2)}} : C^{(2)} \to C,\ s^+_{\Lambda^+})$ be an object of $\overline{\mathcal{M}}_{g,\,n+1}$ where $C^{(2)} = (C \times_S C)^{\mathrm{st}}$ for an object $(\pi_{C\times_S C} : C \times_S C \to C,\ s^+_\Lambda,\ \Delta_C)$ of $\overline{\mathcal{C}}_{g,\,n}$. It is a stable Λ^+-pointed curve over C. We denote the additional section by $\Delta_C = s^+_{n+1} : C \to C^{(2)} \setminus \Sigma_{C^{(2)}}$. We denote by $J^{(m)}(C/S)$ the pull-back of a jet bundle $J^m(C^{(2)} \setminus \Sigma_{C^{(2)}}/C)$ with respect to $\Delta_C : C \to C^{(2)} \setminus \Sigma_{C^{(2)}}$. The restriction of $J^{(m)}(C/S)$ to $C \setminus \Sigma$ is isomorphic to $J^m(C \setminus \Sigma/S)$. We set

$$J^{(\infty)}(C/S) = \varprojlim J^{(m)}(C/S).$$

There is a natural connection along the fiber $\nabla^{(\infty)}_{C/S}$ on $J^{(\infty)}(C/S)$.

Definition 4.4. A stable Λ-pointed curve $\mathfrak{C} = (\pi : C \to S,\, s_\Lambda)$ is called a *regular family* if

1. $C,\ S$ are non-singular.
2. The set of critical values of $\pi : C \to S$ is a normal-crossing divisor on S.

Let V be a VOA, $\mathfrak{C} = (\pi : C \to S,\, s_\Lambda)$ a regular family of stable Λ-pointed curves. We set $V_{C/S} = J^{(\infty)}(C/S) \times_{\mathrm{Aut}\,\mathbb{C}[[z]]} V$, which is the inductive limit of an increasing sequence of finite-dimensional vector bundles over C

$$F_pV_{C/S} = J^{(\infty)}(C/S) \times_{\mathrm{Aut}\,\mathbb{C}[[z]]} F_pV = J^{(p+1)}(C/S) \times_{\mathrm{Aut}(\mathbb{C}[[z]]/(z^{p+2}))} F_pV.$$

The sheaf $\mathcal{O}_C(V_{C/S})$ of sections of $V_{C/S}$ is defined to be the inductive limit $\varinjlim \mathcal{O}_C(F_pV_{C/S})$.

Then $F_0 V_{C/S} \cong C \times V[0]$ is a trivial bundle. The vacuum state $|0\rangle \in V[0]$ induces a global section $|0\rangle_{C/S}$ of $F_0 V_{C/S}$.

The operator

$$\partial_t + L_{-1} : V \otimes \mathbb{C}[[t]] \to V \otimes \mathbb{C}[[t]]$$

and $\nabla^{(\infty)}_{C/S}$ induce a connection along the fiber

$$\nabla^V_{C/S} : \mathcal{O}_C(V_{C/S}) \to \mathcal{O}_C(V_{C/S}) \otimes \omega_{C/S}$$

where $\omega_{C/S}$ is the relative dualizing sheaf. We set

$$\mathrm{Lie}_C(V_{C/S}(*s_\Lambda)) = \frac{\mathcal{O}_C(V_{C/S})(*s_\Lambda) \otimes \omega_{C/S}}{\nabla^V_{C/S} \mathcal{O}_C(V_{C/S})(*s_\Lambda)}.$$

This is a $\pi_C^{-1}\mathcal{O}_S$-Lie algebra.

Let $M_\Lambda = \{M_\alpha\}_{\alpha \in \Lambda}$ is a family of meromorphic V-modules. We consider $\mathcal{O}_{J^{(1)}(C/S)}$-modules

$$(M_\alpha)_{J^{(1)}(C/S)} = \mathcal{O}_{J^{(1)}(C/S)}(J^{(\infty)}(C/S) \times_{\mathrm{Aut}_0 \mathbb{C}[[t]]} M_\alpha).$$

We choose a lift $s^{(1)}_\Lambda = \{s^{(1)}_\alpha S \to J^{(1)}(C/S)\}_{\alpha \in \Lambda}$ of s_Λ. The $\mathcal{O}_S$-Lie algebra

$$\pi_{C*}\mathrm{Lie}_C(V_{C/S}(*s_\Lambda)) = \frac{\pi_{C*}(\mathcal{O}_C(V_{C/S})(*s_\Lambda) \otimes \omega_{C/S})}{\nabla^V_{C/S} \pi_{C*}\mathcal{O}_C(V_{C/S})(*s_\Lambda)}.$$

acts on an $\mathcal{O}_S$-module $\bigotimes_{\alpha \in \Lambda} {s^{(1)}_\alpha}^* (M_\alpha)_{J^{(1)}(C/S)}$.

In general, for a Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-module M, we call $M/(\mathfrak{g} \cdot M)$ the *coinvariant* of $(\mathfrak{g},\, M)$.

Definition 4.5. We denote by $\mathrm{Cov}\,(C/S,\ s^{(1)}_\Lambda \mid V,\ M_\Lambda)$ the coinvariant of

$$\left(\pi_{C*}\mathrm{Lie}_C(V_{C/S}(*s_\Lambda)),\ \bigotimes_{\alpha \in \Lambda} {s^{(1)}_\alpha}^* (M_\alpha)_{J^{(1)}(C/S)}\right),$$

called *sheaf of covacua* over S. The $\mathcal{O}_S$-module

$$\mathrm{Conf}\,(C/S,\ s^{(1)}_\Lambda \mid V,\ M_\Lambda) = \mathcal{H}om_{\mathcal{O}_S}(\mathrm{Cov}\,(C/S,\ s^{(1)}_\Lambda \mid V,\ M_\Lambda),\ \mathcal{O}_S)$$

is called *sheaf of conformal blocks.*

Let $D \subset S$ be the divisor over which each fiber is a singular curve.

Theorem 4.6. *Assume V is C_2-cofinite and that M_Λ is a family of meromorphic V-modules.*
(1) $\mathrm{Cov}\,(C/S,\ s^{(1)}_\Lambda \mid V,\ M_\Lambda)$, $\mathrm{Conf}\,(C/S,\ s^{(1)}_\Lambda \mid V,\ M_\Lambda)$ *are coherent $\mathcal{O}_S$-modules.*
(2) The restriction to $S \smallsetminus D$ of these two $\mathcal{O}_S$-sheaves are locally-free.

The statemet (2) is implied by the exisitence of *KZ connections* explained below.

§ 4.2. The Kodaira-Spencer map and the KZ connection

For a complex manifold X, a divisor D on X and an $\mathcal{O}_C$-module $\mathcal{F}$, we put

$$\begin{aligned}\Omega_X(\log D) &= \{\alpha \in \Omega_X(D) \mid \mathrm{d}\alpha \in \Omega_X^2(D)\},\\ \mathcal{F}(*D) &= \varinjlim \mathcal{F}(mD) \quad (m \in \mathbb{N}).\end{aligned}$$

Let $\Lambda = \{1, \ldots, n\}$. We apply a left exact functor $(-)^\vee = \mathcal{H}om_{\mathcal{O}_C}(-, \mathcal{O}_C)$ to the short exact sequence

$$0 \to \pi_C^*\Omega_S \to \Omega_C(\log s_\Lambda^{\otimes m+1}) \to \Omega_{C/S}((m+1)\cdot s_\Lambda) \to 0 \tag{4.1}$$

where s_Λ denote the divisor induced by the sections $s_i : S \to C$ $(i \in \Lambda)$ and $m \in \mathbb{Z}_{\geq 0}$. Since the support of $\mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_{C/S}((m+1)\cdot s_\Lambda),\ \mathcal{O}_C)$ is contained in Σ_C, it holds $\mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_{C/S}((m+1)\cdot s_\Lambda),\ \mathcal{O}_C) = \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_{C/S},\ \mathcal{O}_C)$. Then we obtain exact sequences of $\mathcal{O}_C$-modules

$$\begin{aligned}&0 \to \Theta_{C/S}(-(m+1)\cdot s_\Lambda) \to \Theta_C(-\log s_\Lambda^{\otimes m+1}) \to \pi_C^*\Theta_S\\ &\quad \to \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_{C/S},\ \mathcal{O}_C) \to 0,\\ &0 \to \Theta_{C/S}(-(m+1)\cdot s_\Lambda) \to \Theta_C(-\log s_\Lambda^{\otimes m+1}) \to \mathcal{M} \to 0,\\ &0 \to \mathcal{M} \to \pi_C^*\Theta_S \to \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_{C/S},\ \mathcal{O}_C) \to 0.\end{aligned}$$

Since π_C is proper, it holds $\pi_{C*}\pi_C^*\Theta_S = \Theta_S$, $R^1\pi_{C*}\pi_C^*\Theta_S = 0$.
Since $\pi_C : C \to S$ is a stable Λ-pointed curve, $\pi_{C*}\Theta_{C/S}((m+1)\cdot s_\Lambda) = 0$.
Thus we obtain exact sequences of $\mathcal{O}_S$-modules

$$\begin{aligned}&0 \to \pi_{C*}\Theta_C(-\log s_\Lambda^{\otimes m+1}) \to \pi_{C*}\mathcal{M} \xrightarrow[\rho_{C/S}]{} R^1\pi_{C*}\Theta_{C/S}(-(m+1)\cdot s_\Lambda)\\ &\quad \to R^1\pi_{C*}\Theta_C(-\log s_\Lambda^{\otimes m+1}) \to R^1\pi_{C*}\mathcal{M} \to 0,\\ &0 \to \pi_{C*}\mathcal{M} \to \Theta_S \to \pi_{C*}\mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_{C/S},\ \mathcal{O}_C) \to R^1\pi_{C*}\mathcal{M} \to 0.\end{aligned}$$

Lemma 4.7. $\pi_{C*}\mathcal{M} \cong \Theta_S(-\log D)$, $R^1\pi_{C*}\mathcal{M} = 0$.

The map $\rho^m_{C/S} : \Theta_S(-\log D) \to R^1\pi_{C*}\Theta_{C/S}(-(m+1)\cdot s_\Lambda)$ is called a *Kodaira-Spencer map* for curves with m-framed marked points. The kernel and the cokernel of it are isomorphic to

$$\pi_{C*}\Theta_C(-\log s_\Lambda^{\otimes m+1}), \quad R^1\pi_{C*}\Theta_C(-\log s_\Lambda^{\otimes m+1})$$

respectively. Applying the functor π_{C*} to the exact sequence

$$0 \to \Theta_{C/S}(-(m+1)\cdot s_\Lambda) \to \Theta_{C/S}(*s_\Lambda) \to \frac{\Theta_{C/S}(*s_\Lambda)}{\Theta_{C/S}(-(m+1)\cdot s_\Lambda)} \to 0$$

we obtain an exact sequnece

$$0 \to \pi_{C*}\Theta_{C/S}(*s_\Lambda) \to \mathcal{L}_S^{\Lambda,\,m} \to \mathcal{R}_S^{\Lambda,\,m} \to 0$$

where

$$\mathcal{L}_S^{\Lambda,\,m} = \pi_{C*}\frac{\Theta_{C/S}(*s_\Lambda)}{\Theta_{C/S}(-(m+1)\cdot s_\Lambda)}, \quad \mathcal{R}_S^{\Lambda,\,m} = R^1\pi_{C*}\Theta_{C/S}(-(m+1)\cdot s_\Lambda).$$

It means that the germs at the marked points of meromorphic vector fields along the fiber generate the deformation of pointed curves.

Let $\mathcal{L}_S^{\Lambda\,\infty} = \varprojlim \mathcal{L}_S^{\Lambda,\,m}$, $\mathcal{R}_S^{\Lambda,\,\infty} = \varprojlim \mathcal{R}_S^{\Lambda,\,m}$. Then we obtain an exact sequnece

$$0 \to \pi_{C*}\Theta_{C/S}(*s_\Lambda) \to \mathcal{L}_S^{\Lambda,\,\infty} \to \mathcal{R}_S^{\Lambda,\,\infty} \to 0.$$

and the Kodaira-Spencer map $\rho_{C/S}^\infty : \Theta_S(-\log D) \to \mathcal{R}_S^{\Lambda,\,\infty}$. Then

$$\mathcal{L}_S^{\Lambda,\,\infty} \cong \bigoplus_{i\in\Lambda} s_i^{-1}(\Theta_{C/S}(*s_i) \otimes_{\mathcal{O}_C} \hat{\mathcal{O}}_{C,\,s_i})$$

is an $\mathcal{O}_S$-Lie algebra.

Since $\mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_{C/S},\, \mathcal{O}_C(*s_\Lambda)) = 0$, we have an exact sequence

$$0 \to \Theta_{C/S}(*s_\Lambda) \to \Theta_C(*s_\Lambda) \to (\pi_C^*\Theta_S)(*s_\Lambda) \to 0.$$

Remark that $\pi_C^{-1}\Theta_S(-\log D)$ is a $\pi_C^{-1}\mathcal{O}_S$-subsheaf of $(\pi_C^*\Theta_S)(*s_\Lambda)$. Let $\mathcal{L}_{C/S}$ be the fiber product of

$$\begin{array}{ccc} & & \pi_C^{-1}\Theta_S(-\log D) \\ & & \downarrow \\ \Theta_C(*s_\Lambda) & \to & \pi_C^*\Theta_S(*s_\Lambda). \end{array}$$

This is a $\mathbb{C}_C$-Lie subalgebra of $\Theta_{C/S}(*s_\Lambda) + \Theta_C$. We have an exact sequence of $\pi_C^{-1}\mathcal{O}_S$-modules

$$0 \to \Theta_{C/S}(*s_\Lambda) \to \mathcal{L}_{C/S} \to \pi_C^{-1}\Theta_S(-\log D) \to 0.$$

Each term is a $\mathbb{C}_C$-Lie algebra, but $\mathcal{L}_{C/S} \to \pi_C^{-1}\Theta_S(-\log D)$ is not a Lie algebra homomorphism.

Since $R^1\pi_{C*}\Theta_{C/S}(*s_\Lambda) = 0$, we obtain an exact sequnece of $\mathcal{O}_S$-modules

$$0 \to \pi_{C*}\Theta_{C/S}(*s_\Lambda) \to \pi_{C*}\mathcal{L}_{C/S} \to \Theta_S(-\log D) \to 0.$$

At $x = s_i(b)$ $(b \in S)$ we have a direct sum decomposition of the tangent space $T_xC = T_xC_b \oplus T_xs_i(S)$. It induces an $\mathcal{O}_S$-homomorphism

$$\widetilde{\rho}_{C/S}^\infty : \pi_{C*}\mathcal{L}_{C/S} \to \mathcal{L}_S^{\Lambda,\,\infty}$$

and a commutative diagram of $\mathcal{O}_S$-modules

$$\begin{array}{ccccccccc} 0 \to & \pi_{C*}\Theta_{C/S}(*s_\Lambda) & \to & \pi_{C*}\mathcal{L}_{C/S} & \to & \Theta_S(-\log D) & \to 0 \\ & \downarrow \mathrm{id} & & \downarrow \widetilde{\rho}^{\infty}_{C/S} & & \downarrow \rho^{\infty}_{C/S} & \\ 0 \to & \pi_{C*}\Theta_{C/S}(*s_\Lambda) & \to & \mathcal{L}^{\Lambda,\infty}_S & \to & \mathcal{R}^{\Lambda,\infty}_S & \to 0. \end{array}$$

Lemma 4.8. *$\widetilde{\rho}^{\infty}_{C/S}$ is a Lie algebra homomorphism.*

Let $\mathcal{V}ir^{\Lambda}_S(c) = \mathcal{L}^{\Lambda,\infty}_S \oplus \mathcal{O}_S$ be the central extention of the $\mathcal{O}_S$-Lie algebra $\mathcal{L}^{\Lambda,\infty}_S$ with central charge $c \in \mathbb{C}$. It acts naturally on

$$\mathcal{M}_\Lambda = \bigotimes_{\alpha\in\Lambda} s^{(1)*}_\alpha (M_\alpha)_{J^{(1)}(C/S)}.$$

Remark that $|0\rangle_{C/S}$ acts trivially on $\mathrm{Lie}_C(V_{C/S}(*s_\Lambda))$ and that $|\mathrm{vir}\rangle$ induces a global section of $F_2V_{C/S}/F_0V_{C/S}$ along the fiber. Hence $\mathcal{V}ir^{\Lambda}_S(c)$ acts on $\mathfrak{g} = \pi_{C*}\mathrm{Lie}_C(V_{C/S}(*s_\Lambda))$. These actions are compatible with the action of $\mathfrak{g}$ on $\mathcal{M}_\Lambda$. Thus we obtain actions of $\mathcal{V}ir^{\Lambda}_S(c)$ on

$$\mathrm{Cov}\,(C/S,\ s^{(1)}_\Lambda \mid V,\ M_\Lambda), \quad \mathrm{Conf}\,(C/S,\ s^{(1)}_\Lambda \mid V,\ M_\Lambda).$$

The restrictions to $\pi_{C*}\Theta_{C/S}(*s_\Lambda)$ are trivial actions.

Let $\pi_{C*}\mathcal{L}_{C/S} \oplus \mathcal{O}_S \to \pi_{C*}\mathcal{L}_{C/S}$ be the central extension that is the pull-back of $\mathcal{V}ir^{\Lambda}_S(c) \to \mathcal{L}^{\Lambda,\infty}_S$ by $\widetilde{\rho}^{\infty}_{C/S}$. We set

$$\widetilde{\Theta}_S(-\log D) = \frac{\pi_{C*}\mathcal{L}_{C/S} \oplus \mathcal{O}_S}{\pi_{C*}\Theta_{C/S}(*s_\Lambda)},$$

which is a central extension of $\Theta_S(-\log D)$. Now we obtain the following.

Theorem 4.9. *$\widetilde{\Theta}_S(-\log D)$ acts naturally on*

$$\mathrm{Cov}\,(C/S,\ s^{(1)}_\Lambda \mid V,\ M_\Lambda), \quad \mathrm{Conf}\,(C/S,\ s^{(1)}_\Lambda \mid V,\ M_\Lambda).$$

The connections on those two sheaves induced by this theorem are called *KZ connections.*

§ 5. Factorization

Let $S = \mathrm{Spec}\,\mathbb{C}[[t]]$, $D = \{t = 0\} \subset S$, $\Lambda = \{1, \dots, n\}$, and $(\pi : C \to S,\ s_\Lambda)$ be a regular family of Λ-pointed stable curves. We identify $\mathcal{O}_S$-modules on S with $\mathbb{C}[[t]]$-modules. Assume the singular fiber $C_0 = \pi^{-1}(D)$ has exactly one double point P_0. Let $\widetilde{C}_0 \to C_0$ be the normalization, $s^+,\ s^- : D \to J^{(1)}(\widetilde{C}_0/D)$ the inverse image of the double point with 1-framings, and $P^+,\ P^-$ the inverse image of the double point P_0.

Let x, y be a local coordinate of C around P_0 such that $\widehat{\mathcal{O}}_{C,\,P_0} = \mathbb{C}[[x,\,y]]$, $t = x\,y$. (For simplicity, we denote by t the pull-back of $t \in \mathcal{O}_S$ by π.) Assume that x, y give local coordinates $\widetilde{x}$, $\widetilde{y}$ around P^+, P^-, respectively.

The relative dualizing sheaf of $\pi : C \to S$ is described as

$$\widehat{\omega}_{C/S,\,P_0} = \mathbb{C}[[x,\,y]] \cdot \frac{\mathrm{d}x}{x}.$$

Remark that $\dfrac{\mathrm{d}x}{x} + \dfrac{\mathrm{d}y}{y} = 0$ in $\widehat{\omega}_{C/S,\,P_0}$. On the other hand,

$$\widehat{\omega}_{\widetilde{C}_0/D,\,P^+} = \mathbb{C}[[\widetilde{x}]] \cdot \mathrm{d}\widetilde{x}, \quad \widehat{\omega}_{\widetilde{C}_0/D,\,P^-} = \mathbb{C}[[\widetilde{y}]] \cdot \mathrm{d}\widetilde{y}.$$

.

At the point $(P_0,\,P_0)$ of $C \times_S C$, it holds

$$\widehat{\mathcal{O}}_{C\times_S C,\,(P_0,\,P_0)} \cong \mathbb{C}[[x_0,\,y_0,\,x_1,\,y_1]]/(x_0\,y_0 - x_1\,y_1), \quad t = x_0\,y_0 = x_1\,y_1.$$

Let

$$X \to \operatorname{Spec}\mathbb{C}[[x_0,\,x_1]], \quad Y \to \operatorname{Spec}\mathbb{C}[[y_0,\,y_1]]$$

be the blowing-up at the $x_0 = x_1 = 0$, $y_0 = y_1 = 0$, respectively. For the stabilization $\sigma : C^{(2)} = (C \times_S C)^{\mathrm{st}} \to C \times_S C$, the infinitesimal neighborood of $E = \sigma^{-1}(P_0,\,P_0) \subset C^{(2)}$ is embedded in $X \times_{\operatorname{Spec}\mathbb{C}} Y$. The defining equation of the image is $x_0 y_0 = x_1 y_1$.

We set $x_1 = (1+\xi)\,x_0$, $y_1 = (1+\eta)\,y_0$. It holds $(1+\xi)(1+\eta) = 1$. The defining equation of the diagonal, the image of $\Delta : C \to C^{(2)}$, is $\xi = 0$, or $\eta = 0$.

We fix a local isomorphism around P_0 between V_C and $\displaystyle\bigoplus_{\Delta=0}^{\infty} V[\Delta] \otimes \omega_{C/S}^{\otimes(-\Delta)}$.

The local isomorphisms around P_0 from $\omega_{C/S}(V_{C/S})$ to $V \otimes \omega_{C/S}$ corresponding to ξ, η are denoted by φ_ξ, φ_η. For $a \in V$ it holds

$$\varphi_\eta \circ \varphi_\xi^{-1}\left(a \otimes x^m\,y^n \cdot \frac{\mathrm{d}x}{x}\right) = \mathrm{e}^{L_1}(-1)^{L_0} a \otimes x^m\,y^n \cdot \left(-\frac{\mathrm{d}y}{y}\right).$$

For $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ we can define the sheaf of covacua

$$\operatorname{Cov}(\widetilde{C}_0/D,\ s_\Lambda^{(1)},\ s^\pm \mid V;\ M_\Lambda,\ B^m),$$

which is a free $\mathbb{C}[[t]]/(t^{m+1})$-module. The $U(V)$-bimodule B^m is considered to be a left $U(V) \otimes U(V)$-module by way of θ.

The germs of $\mathrm{Lie}_{\widetilde{C}_0}(V_{\widetilde{C}_0/D}(*\{s_\Lambda,\,s^\pm\}))$ at P^+, P^- corresponding to

$$\varphi_\xi^{-1}\left(a \otimes x^m\,y^n \cdot \frac{\mathrm{d}x}{x}\right) = \varphi_\eta^{-1}\left(\mathrm{e}^{L_1}(-1)^{L_0} a \otimes x^m\,y^n \cdot \left(-\frac{\mathrm{d}y}{y}\right)\right)$$

are $t^n\, J_{m-n}(a),\ -t^m\,\theta(J_{m-n}(a)) \in \mathrm{Lie}(V)$, obtained by

$$a \otimes \widetilde{x}^m \left(\frac{t}{\widetilde{x}}\right)^n \cdot \left(\frac{\mathrm{d}\widetilde{x}}{\widetilde{x}}\right)^{1-\Delta}, \quad -\mathrm{e}^{L_1}(-1)^{L_0} a \otimes \left(\frac{t}{\widetilde{y}}\right)^m \widetilde{y}^n \cdot \left(\frac{\mathrm{d}\widetilde{y}}{\widetilde{y}}\right)^{1-\Delta},$$

respectively, by way of appropriate local isomorphisms around P^+, P^- between $V_{\widetilde{C}_0/D}$ and $\bigoplus_{\Delta=0}^{\infty} V[\Delta] \otimes \omega_{\widetilde{C}_0/D}^{\otimes(-\Delta)}$.

Remark that

$$t^n\, J_{m-n}(a) \cdot e_\infty(t) + e_\infty(t) \cdot (-t^m\, J_{m-n}(a)) = 0.$$

We denote this relation by $J_{m-n}(a) \cdot e_\infty(t) \sim e_\infty(t) \cdot J_{m-n}(a)$.

Let $X \in \pi_{C*}\mathrm{Lie}_C(V_{C/S}(*s_\Lambda))$. Assume the expansion of X at P_0 is given by

$$\varphi_\xi^{-1}\left(\sum_{m,\,n\ge 0} a_{m,\,n} \otimes x^m\, y^n \cdot \frac{\mathrm{d}x}{x}\right) \quad (a_{m,\,n} \in V).$$

Let $\widetilde{X} \in \pi_{C_0*}\mathrm{Lie}_{C_0}(V_{C_0/D}(*\{s_\Lambda,\, s^\pm\}))\otimes\mathbb{C}[[t]]$ be the induced element whose expansions at P^+, P^- correspond to

$$\sum_{m,\,n\ge 0} t^n\, J_{m-n}(a_{m,\,n}), \quad -\sum_{m,\,n\ge 0} t^m\,\theta(J_{m-n}(a_{m,\,n})),$$

respectively,

We fix local coordinates of the fibers of $\pi : C \to S$ around the marked points s_Λ. We identify the $\mathcal{O}_S$-module

$$\mathcal{M}_\Lambda = \bigotimes_{\alpha\in\Lambda} s_\alpha^{(1)*}(M_\alpha)_{J^{(1)}(C/S)}$$

with $\bigotimes_{\alpha\in\Lambda} M_\alpha \otimes \mathbb{C}[[t]]$.

For $n \in \mathbb{Z}$ the map

$$\mathcal{M}_\Lambda \to \mathcal{M}_\Lambda \otimes_{\mathbb{C}[[t]]} B^n, \quad u_\Lambda \mapsto u_\Lambda \otimes e_n(t)$$

induces a $\mathbb{C}[[t]]$-homomorphism

$$F^n : \mathrm{Cov}\,(C/S,\ s_\Lambda^{(1)} \mid V,\ M_\Lambda) \to \mathrm{Cov}\,(\widetilde{C}_0/D,\ s_\Lambda^{(1)},\, s^\pm \mid V;\ M_\Lambda,\, B^n).$$

Theorem 5.1. *(Y. H.) The $\mathbb{C}[[t]]$-homomorphism*

$$\begin{aligned} F^\infty = \varprojlim F^n : \mathrm{Cov}\,&(C/S,\ s_\Lambda^{(1)} \mid V,\ M_\Lambda) \\ &\to \varprojlim \left\{\mathrm{Cov}\,(\widetilde{C}_0/D,\ s_\Lambda^{(1)},\, s^\pm \mid V;\ M_\Lambda,\, B^n)\right\}_n \end{aligned}$$

is surjective and the kernel is a torsion $\mathbb{C}[[t]]$-module.

Proof. We set $\mathcal{L} = \mathrm{Lie}_{\widetilde{C}_0}(V_{\widetilde{C}_0/D}(*\{s_\Lambda,\ s^\pm\})) \otimes \mathbb{C}[[t]]$.

Let $u_\Lambda \otimes X \cdot e_n(t) \cdot Y \in \mathcal{M}_\Lambda \otimes_{\mathbb{C}[[t]]} B^n,\ X,\ Y \in \mathrm{Lie}(V)$.

There exists $Y_{\widetilde{C}_0/D} \in \mathcal{L}$ such that $Y_{\widetilde{C}_0/D} \cdot (X \cdot e_n(t)) = 0$ at P^+ and $e_n(t) \cdot Y_{\widetilde{C}_0/D} = e_n(t) \cdot \theta(Y)$ at P^-. Hence $u_\Lambda \otimes X \cdot e_n(t) \cdot \theta(Y)$ is equivalent to $u'_\Lambda \otimes X \cdot e_n(t)$. There exists $X_{\widetilde{C}_0/D} \in \mathcal{L}$ such that $X_{\widetilde{C}_0/D} \cdot e_n(t) = X \cdot e_n(t)$ at P^+, and $e_n(t) \cdot \theta(X_{\widetilde{C}_0/D}) = 0$ at P^-. Hence $u'_\Lambda \otimes X \cdot e_n(t)$ is equivalent to $u''_\Lambda \otimes e_n(t)$. Hence F^n is surjective. It implies that F^∞ is surjective since $\mathrm{Cov}\,(C/S,\ s_\Lambda^{(1)} \mid V,\ M_\Lambda)$ is a finitely generated $\mathbb{C}[[t]]$-module.

We fix $n \in \mathbb{N}$. Let $X_{\widetilde{C}_0/D} \in \mathcal{L}, \quad w_\Lambda \otimes W \in \mathcal{M}_\Lambda \otimes B^n$.

There exisits a decomposition $X_{\widetilde{C}_0/D} = X^+_{\widetilde{C}_0/D} + X^-_{\widetilde{C}_0/D}$ such that $e_n(t) \cdot \theta(X^+_{\widetilde{C}_0/D}) = 0$ at P^- and $X^-_{\widetilde{C}_0/D} \cdot e_n(t) = 0$ at P^+. For some $N \in \mathbb{N}$ and $Y^+,\ Y^- \in \mathrm{Lie}(V) \otimes \mathbb{C}[[t]]$ it holds

$$t^N\, W = Y^+ \cdot e_n(t) = e_n(t) \cdot \theta(Y^-).$$

There exists $Y^+_{\widetilde{C}_0/D} \in \mathcal{L}$ such that $Y^+_{\widetilde{C}_0/D} \cdot e_n(t) = Y^+ \cdot e_n(t)$ at P^+ and $e_n(t) \cdot \theta(Y^+_{\widetilde{C}_0/D}) = 0$ at P^-.

Thus

$$\begin{aligned}
&t^N\, X_{\widetilde{C}_0/D} \cdot (w_\Lambda \otimes W) \\
&= X^+_{\widetilde{C}_0/D} \cdot (w_\Lambda \otimes Y^+ \cdot e_n(t)) + X^-_{\widetilde{C}_0/D} \cdot (w_\Lambda \otimes e_n(t) \cdot \theta(Y^-)).
\end{aligned}$$

There exist $Z^+_{\widetilde{C}_0/D},\ Z^-_{\widetilde{C}_0/D} \in X_{\widetilde{C}_0/D} \in \mathrm{Lie}_{\widetilde{C}_0}(V_{\widetilde{C}_0/D}(*\{s_\Lambda,\ s^\pm\}))$ such that

$$Z^+_{\widetilde{C}_0/D} \cdot e_n(t) = X^+_{\widetilde{C}_0/D} \cdot (Y^+ \cdot e_n(t)), \quad e_n(t) \cdot \theta(Z^+_{\widetilde{C}_0/D}) = 0,$$

$$e_n(t) \cdot \theta(Z^-_{\widetilde{C}_0/D}) = (e_n(t) \cdot Y^-) \cdot X^-_{\widetilde{C}_0/D}, \quad Z^-_{\widetilde{C}_0/D} \cdot e_n(t) = 0.$$

We put $Z_{\widetilde{C}_0/D} = Z^+_{\widetilde{C}_0/D} + Z^-_{\widetilde{C}_0/D}$. Then

$$\begin{aligned}
&t^N\, X_{\widetilde{C}_0/D} \cdot (w_\Lambda \otimes W) - Z_{\widetilde{C}_0/D} \cdot (w_\Lambda \otimes e_n(t)) \\
&= X_{\widetilde{C}_0/D} \cdot w_\Lambda \otimes Y^+ \cdot e_n(t) - Z_{\widetilde{C}_0/D} \cdot w_\Lambda \otimes e_n(t) \\
&= Y^+_{\widetilde{C}_0/D} \cdot (X_{\widetilde{C}_0/D} \cdot w_\Lambda \otimes e_n(t)) \\
&\quad - (Y^+_{\widetilde{C}_0/D} \cdot (X_{\widetilde{C}_0/D} \cdot w_\Lambda) + Z_{\widetilde{C}_0/D} \cdot w_\Lambda) \otimes e_n(t).
\end{aligned}$$

There exists $T_{\widetilde{C}_0/D} \in \mathcal{L}$ such that

$$T_{\widetilde{C}_0/D} \cdot w_\Lambda = Y^+_{\widetilde{C}_0/D} \cdot (X_{\widetilde{C}_0/D} \cdot w_\Lambda) + Z_{\widetilde{C}_0/D} \cdot w_\Lambda,$$

and $T_{\widetilde{C}_0/D} \cdot e_n(t) + e_n(t) \cdot \theta(T_{\widetilde{C}_0/D}) = 0$. Hence there exist $X'_{\widetilde{C}_0/D} \in \mathcal{L}$ and $w'_\Lambda \in \mathcal{M}_\Lambda$ such that

$$t^N\, X_{\widetilde{C}_0/D} \cdot (w_\Lambda \otimes W) = X'_{\widetilde{C}_0/D} \cdot (w'_\Lambda \otimes e_n(t)).$$

Let $v_\Lambda \in \mathcal{M}_\Lambda$. If $F^\infty([v_\Lambda]) = 0$, then for any n there exist $K,\, k \in \mathbb{N}$ and

$$X_{\widetilde{C}_0/D,\, i} \in \mathcal{L}, \quad w_{\Lambda,\, i} \in \mathcal{M}_\Lambda \quad (i = 1,\, \ldots,\, k)$$

such that

$$t^K\, v_\Lambda \otimes e_n(t) = \sum_{i=1}^{k} X_{\widetilde{C}_0/D,\, i} \cdot (w_{\Lambda,\, i} \otimes e_n(t))$$

and $w_{\Lambda,\, i}$ $(i = 1,\, \ldots,\, k)$ are linearly independent. For each i there exists a decomposition $X_{\widetilde{C}_0/D,\, i} = X^0_{\widetilde{C}_0/D,\, i} + X^1_{\widetilde{C}_0/D,\, i}$ such that

$$X^0_{\widetilde{C}_0/D,\, i} \in \Gamma(\widetilde{C}_0,\; V[0] \otimes \omega_{\widetilde{C}_0/D}(*\{s_\Lambda,\, s^\pm\})) \otimes \mathbb{C}[[t]],$$

$$X^1_{\widetilde{C}_0/D,\, i} \cdot e_n(t) + e_n(t) \cdot \theta(X^1_{\widetilde{C}_0/D,\, i}) = 0.$$

The residue theorem implies $X^0_{\widetilde{C}_0/D,\, i} \cdot (w_{\Lambda,\, i} \otimes e_n(t)) = 0$. Hence

$$\begin{aligned} t^K\, v_\Lambda \otimes e_n(t) &= \sum_{i=1}^{k} X^1_{\widetilde{C}_0/D,\, i} \cdot (w_{\Lambda,\, i} \otimes e_n(t)) \\ &= \left(\sum_{i=1}^{k} X^1_{\widetilde{C}_0/D,\, i} \cdot w_{\Lambda,\, i} \right) \otimes e_n(t). \end{aligned}$$

Therefore $[t^K\, v_\Lambda] = 0$ in $\mathrm{Cov}\,(C/S,\; s^{(1)}_\Lambda \mid V,\; M_\Lambda)$. Hence the kernel of F^∞ is a torsion $\mathbb{C}[[t]]$-module. $\square$

The theorem above shall be studied again in the forthcoming paper with A. Tsuchiya and T. Matsumoto.

§ 6. Fusion tensor products

Let V be a C_2-cofinite VOA. For meromorphic V-modules $M_1,\, M_2$, we set

$$M_1 \otimes_V M_2 = \mathrm{Cov}\,(\mathbb{P}^1,\, \{0,\, 1,\, \infty\} \mid V,\; \{M_1,\, M_2,\, U(V)^{\mathrm{reg}}_\theta\}),$$

where $U(V)^{\mathrm{reg}}_\theta$ means the coinvariant is defined by the right action of $U(V)$ by way of θ. The remaining left action on $U(V)^{\mathrm{reg}}$ induces the left action on $M_1 \otimes_V M_2$. Then $M_1 \otimes_V M_2$ is a meromorphic V-module. For meromorphic V-modules $M_1,\, M_2,\, M_3$, isomorphisms

$$(M_1 \otimes_V M_2) \otimes_V M_3 \cong (M_2 \otimes_V M_3) \otimes_V M_1 \cong (M_3 \otimes_V M_1) \otimes_V M_2$$

are given by the factorization for

$$\mathrm{Cov}\,(\mathbb{P}^1,\, \{0,\, 1,\, t,\, \infty\} \mid V,\; \{M_1,\, M_2,\, M_3,\, U(V)^{\mathrm{reg}}\}).$$

The pentagon relation is proved by the factorization for

$$\mathrm{Cov}\,(\mathbb{P}^1,\ \{0,\ 1,\ s,\ t,\ \infty\} \mid V,\ \{M_1,\ M_2,\ M_3,\ M_4,\ U(V)^{\mathrm{reg}}\}).$$

For a meromorphic V-module M we can obtain a $U(V)$-bimodule

$$M^\beta = \mathrm{Cov}\,(\mathbb{P}^1,\ \{0,\ 1,\ \infty\} \mid V,\ \{U(V)^{\mathrm{reg}},\ M,\ U(V)^{\mathrm{reg}}_\theta\}).$$

It holds $M^\beta \otimes_{U(V)} V \cong M$. The functor given by $M \mapsto M^\beta$ is called *module-bimodule correspondence*. The compatibility of tensor products for this functor is proved by the factorization.

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